Circuit analysis in Laplace domain

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Laplace transform pairs

- As long as f(t) doesn't grow faster than an exponential function, for each f(t) in time domain there is a **unique** F(s) in Laplace domain (frequency domain) and for each F(s) there is a **unique** f(t). f(t) and F(s) are therefore transform pairs.

$$f(t) \Leftrightarrow F(s)$$

- Check Laplace transform table for most popularly used Laplace transform pairs.

Laplace transform pairs

- Laplace transform is a linear operator

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s)$$

$$L[f(t)+g(t)] = L[f(t)] + L[g(t)]$$

$$f(t)+g(t) \Leftrightarrow \quad F(s)+G(s)$$

Laplace transform properties

- Time Integration:

$$L\left[\int_0^t f(x)dx\right] = \frac{1}{s}F(s)$$

- Time Differentiation:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Laplace transform properties

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - sf(0) - f'(0)$$

$$L\left[\frac{d^3 f(t)}{dt^3}\right] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

general case

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

Laplace transform to solve ODEs

- Find y(t)

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 1$$

Laplace domain:

$$s^2Y(s) + 5sY(s) + 6Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s^3 + 5s^2 + 6s}$$

Partial fraction expansion

$$Y(s) = \frac{1}{s^3 + 5s^2 + 6s} = \frac{1}{s(s+2)(s+3)}$$

$$Y(s) = \frac{1}{3(s+3)} - \frac{1}{2(s+2)} + \frac{1}{6s}$$

$$y(t) = \frac{1}{3}e^{-3t} - \frac{1}{2} \cdot e^{-2t} + \frac{1}{6}$$

Matlab command for partial fraction expansion:

b = [1];

a = [1 5 6 0];

[r,p,k] = residue(b,a)

Laplace transform to solve ODEs

- Find y(t)

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 1$$

Laplace domain: $$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s^3 + 3s^2 + 2s}$$

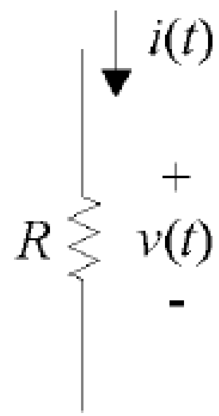
$$Y(s) = \frac{1}{2(s+2)} - \frac{1}{(s+1)} + \frac{1}{2s}$$

Time domain: $$y(t) = \frac{1}{2}e^{-2t} - e^{-t} + \frac{1}{2}$$

Basic components in Laplace domain

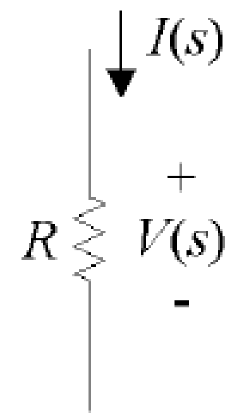
- **Resistor**

<u>Time Domain</u>



$v(t)=R\,i(t)$

<u>Laplace Domain</u>

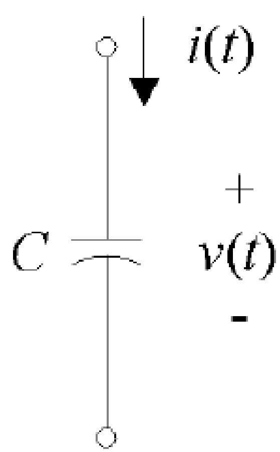


$V(s)=R\,I(s)$

Basic components in Laplace domain

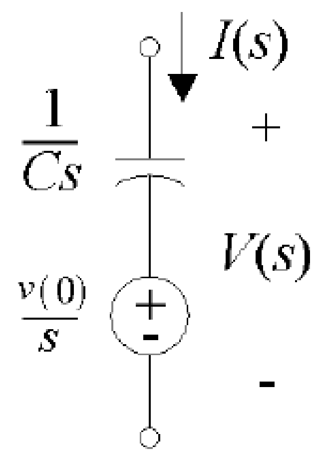
- **Capacitor**

<u>Time Domain</u>



$$v(t) = v(0) + \frac{1}{C}\int_0^t i(t)\,\mathrm{d}t$$

<u>Laplace Domain</u>

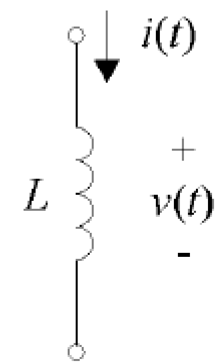


$$V(s) = \frac{v(0)}{s} + \frac{I(s)}{Cs}$$

Basic components in Laplace domain

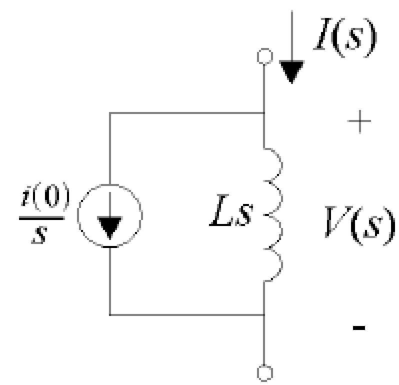
- **Inductor**

Time Domain



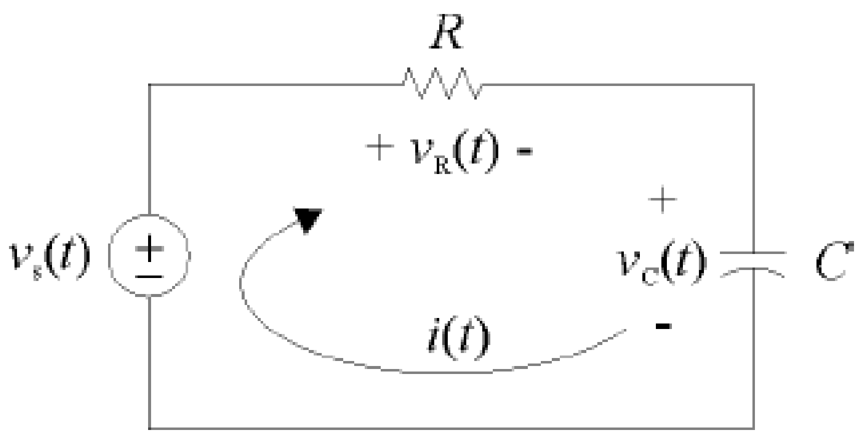
$$v(t) = \mathrm{L}\frac{i(t)}{\mathrm{d}t}\ \text{, I.C.: } i(0)$$

Laplace Domain



$$I(s) = \frac{i(0)}{s} + \frac{V(s)}{L\,s}$$

Circuit analysis in Laplace domain



$$V_s(t) = RC\frac{dV_c(t)}{dt} + V_c(t)$$

$$V_s(t) = u(t)$$

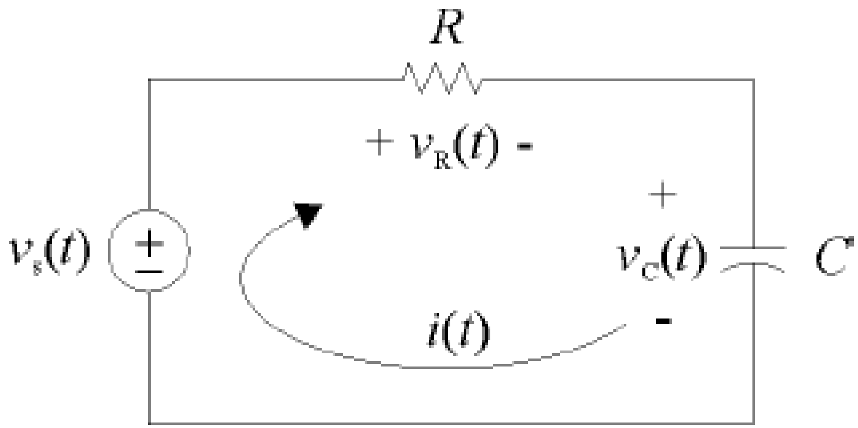
Laplace transform

$$V_c(s) = \frac{1}{1+RCs}\cdot\frac{1}{s}$$

Inverse Laplace transform

$$V_c(t) = 1 - e^{-\frac{t}{RC}}$$

RC circuit



- What if $v_s(t)$ is a delta function?

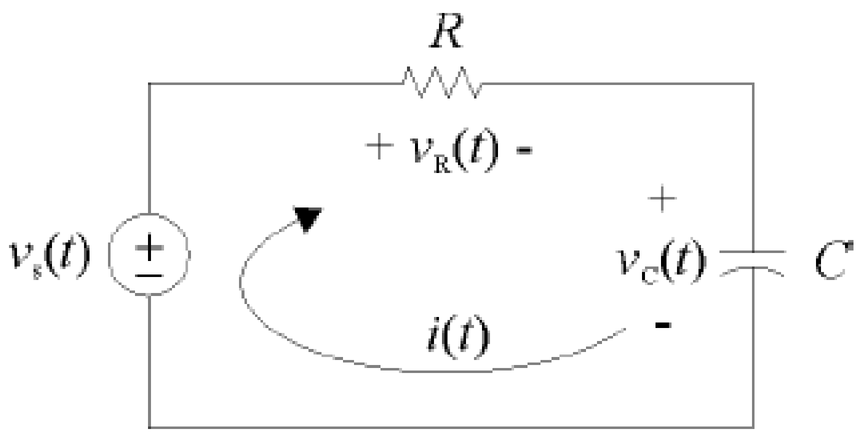
$$v_s(t) = \delta(t)$$

$$RC\frac{dv_c(t)}{dt} = v_s(t) - v_c(t)$$

$$v_c(t) = ?$$

$$V_c(s) = \frac{1}{1 + RCs} \quad \Rightarrow \quad v_c(t) = \frac{1}{RC}e^{-\frac{t}{RC}}$$

RC circuit

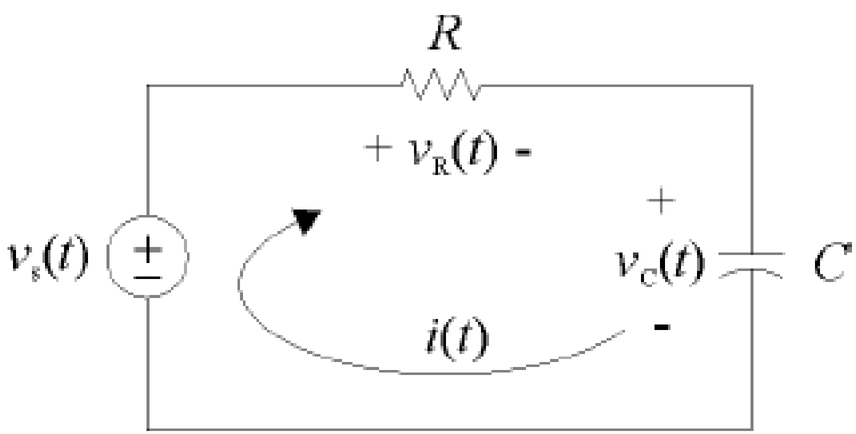


- What is the voltage across the capacitor given that $v_s(t)$ is a sinusoidal signal?

$$v_s(t) = \sin(t)$$

$$v_C(t) = ?$$

RC circuit



- Input signal:

$$v_s(t) = \sin(t)$$

- What is the voltage across the capacitor?

$$V_s(s) = \frac{1}{s^2 + 1}$$

$$V_c(s) = \frac{1}{1 + RCs} \cdot \frac{1}{s^2 + 1}$$

RC circuit

- Response to the 1 rad/s sinusoidal signal

$$V_c(s) = \frac{1/RC}{s + 1/RC} \cdot \frac{1}{s^2 + 1}$$

$$L^{-1}[V_c(s)] = \frac{1}{RC}\left(\frac{1}{(\frac{1}{RC})^2 + 1} e^{-\frac{t}{RC}} + \frac{1}{\sqrt{(\frac{1}{RC})^2 + 1}} \sin(t - \theta)\right)$$

$$= \frac{RC}{(RC)^2 + 1} e^{-\frac{t}{RC}} + \frac{1}{\sqrt{(RC)^2 + 1}} \sin(t - \theta)$$

Where: $\theta = \tan^{-1}(RC)$

Transient response

Steady-state response